Linear Dynamics for Everyone: Part 1

> Why natural frequency analysis is good for you and your design.

BY GEORGE LAIRD

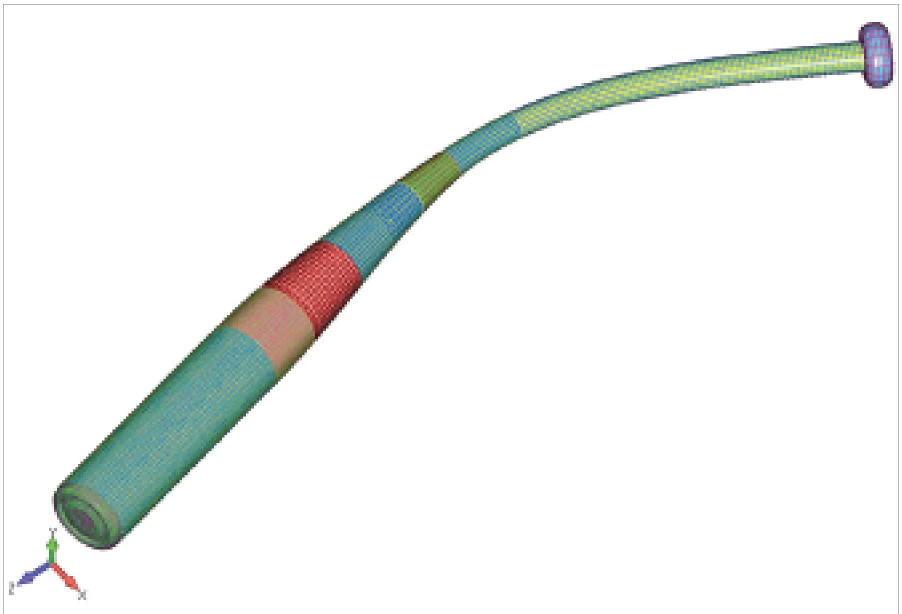

Figure 1: First vibration mode shape for an NCAA aluminum baseball bat is shown here.

Analysis work is rarely done because we have spare time or are just curious about the mechanical behavior of a part or system. It's typically performed because we are worried that the design might fail in a costly or dangerous manner. Depending on the potential failure mode our anxiety might not be too high, but given today's demanding OEMs and litigious public, the task could involve high drama with your name written all over it.

If you've done analysis, you're comfortable with the concepts involved in static stress analysis; you define the loading and boundary conditions, and identify success with a model bathed in soothing tones of gray and blue with nary a red region to be seen. However, in the back of your mind you might wonder about that large vibrating motor or the plant machinery that hums at a constant 12.5Hz. Alternatively, maybe you have an electronics enclosure that is to be mounted on the side of a building in an earthquake-prone region and your boss is questioning your bracket design. Whatever the case, you have the static world under control. What about the rest?

In this series of articles, we'll briefly review dynamic analysis fundamentals and see how they can easily be applied to make sure your design remains strong and rock solid in the face of dynamic events, whether simple vibrations, earthquakes, or even rocket launches.

KEEPING IT SIMPLE

Static stress analysis is the proverbial "walk-in-the-park" for most people doing analysis work. It feels straightforward: we apply a fixed load and examine the re-

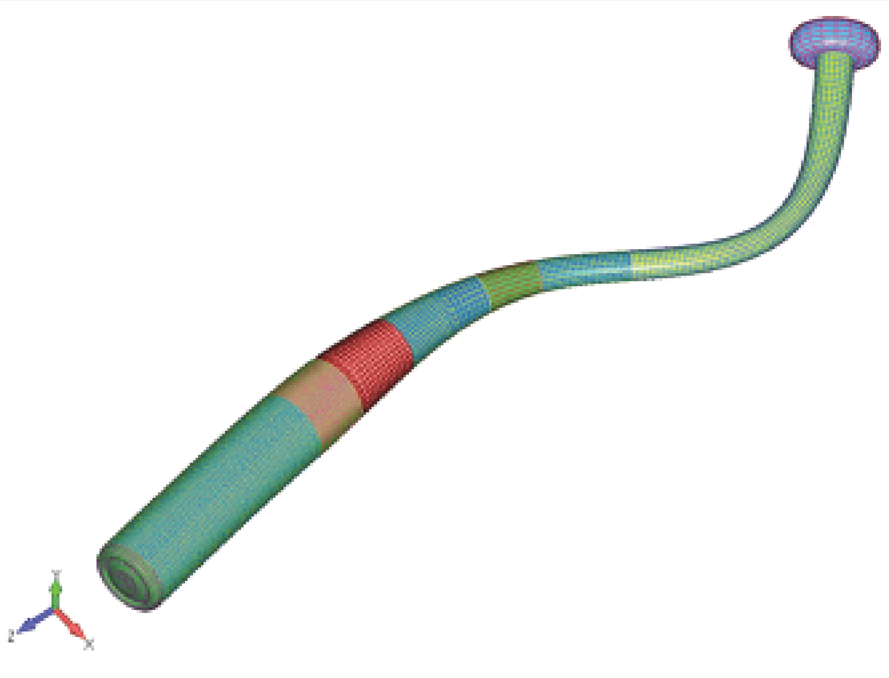

Figure 2: Second vibration mode shape is shown here for an NCAA aluminum baseball bat.

sulting static behavior (generally linear, given linear material behavior). We get back some nice clean stresses and deflections that hopefully match our intuition for how our design should behave. While there might be a few hiccups along the way, the end result usually appears logical to our mechanical minds.

The dynamic behavior of a structure can also be viewed in the same light if we just shift our perspective a bit and think in terms of how our structure should naturally deform during a dynamic event. Whenever a structure is hit or given some sort of time-varying load (transient or steady-state), it will respond to this load with a very characteristic behavior. If the load is not incredibly massive and the structure doesn't blow up or plastically deform as a result, then the dynamic response of your structure will most likely be linear. That is to say, if the load is removed and the structure is given a chance to calm down, then it will return to its undeformed state. This is the same concept to use in linear static stress analysis: when the load is removed the stress in the structure goes back to zero.

What exactly do we mean by characteristic dynamic behavior? All structures have natural or characteristic modes of vibration. The sound or note from a guitar string is all about its natural frequency of vibration. When a guitar string is plucked it will vibrate at a certain note or tone. This note is at the string's characteristic frequency.

Another example is aluminum baseball bats. The best aluminum baseball bats are designed with characteristic vibrations that attempt to limit the sting that occurs when you hit a ball outside the sweet spot on the bat. Each frequency creates a physical deformation or shape, and the total dynamic response of the bat is a combination of all its characteristic mode shapes (*see Figures 1 and 2*).

In finite element analysis (FEA), these natural frequencies are called eigenvalues and their shapes are noted as eigenvectors or eigenmodes. This nomenclature is rooted in German and the word eigen denotes "characteristic" or "peculiar to" and came into common use with mid-19th century mathematicians. With dynamic analyses, you'll also see the terms normal modes and normal modes analysis. The use of the word normal prior to mode is just another way to say natural, characteristic, or eigen. When describing mode shapes, our preference is to just say normal modes since they represent the inherent natural response of the structure.

A BEAM AS ONE EXAMPLE

If we picture a simply supported beam (fixed at one end), its natural mode shapes are determined by its geometry while its frequency of motion is fixed by its stiffness and density. Got all of that? Take a look at the graphic of our beam for its first three modes (*see Figures 3 and 4*). The first three modes of the beam are well-defined but come in pairs to cover all permissible ranges of motion for that beam. In 3D, the first mode can oscillate within a 360-degree envelope around its longitudinal axis. Numerically, the eigen solution process just gives us the two orthogonal modes, but it implies the full 360-degree envelope.

All structures have a nearly infinite number of permissible shapes or eigenvalues/eigenmodes. Fortunately, only the

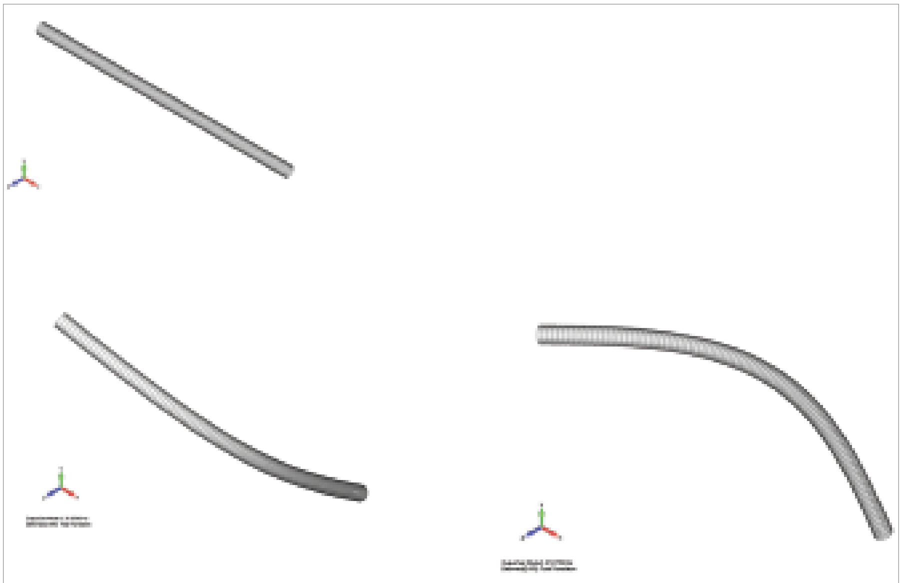

Figure 3: Undisturbed simple beam, plus two of the first vibration mode shapes (two directions of motion).

lower frequencies dominate the response of the structure so we can typically ignore the higher frequencies. A rule of thumb is that the first three modes capture the majority of the response of the structure and therefore one can safely ignore the higher frequencies. (The reasoning for this statement will be given in Part II of this series).

The frequency of these modes or their eigenvalues is dependent upon the stiffness and the density of the beam. The frequency equation for structures can thus be written as:

$$\omega = \sqrt{K / m}$$

where K is the stiffness of the structure and m is the mass. This wonderfully simple equation represents a great deal of information about the system. The classic way to graphically describe this equation is with a mass suspended by a spring, where the mass block can only move up and down or has one degree of freedom (DOF) in FEA parlance. The eigenmode of this system is up and down.

PAPER MILL DESIGN EXAMPLE

In commercially interesting structures, the same equation holds. The eigenvalue of the structure is still determined by

$$\omega = \sqrt{K / m}$$

For example, consider a forming board used within a paper mill. The structure is 10 meters long and made of stainless steel. The paper mill has an operating frequency of around 9Hz. If the structure's natural frequency is near this operating frequency, it will quickly resonate and tear itself apart. More importantly, it will also take the multi-million dollar paper mill along with it (*see Figures 5 and 6*).

The parameters of the original design placed the first mode at 8.4Hz, which would have been a disaster. The forming board is manufactured from 9.5mm-thick stainless-steel plates, so our first design inclination was to simply increase the thickness of the plates. We pursued this approach for several days but as we increased the thickness, the mass of the structure also increased almost in lockstep with the stiffness (see above equation). At the end of all this head banging, we got a marginal improvement (~11Hz resonance) with 25mm-thick plates, but it was going to cost a fortune to manufacture.

At this point we stepped back from our rush to find a solution and thought about how stiffness is developed in long slender structures. We realized that we had very little shear transfer between the top and bottom surfaces of the forming board. This insight led us to add diagonal steel rods that would connect the top and bottom planes and allowed us to keep the thickness of the plates at 9.5mm. The new design tested out on the computer with a first mode frequency of 13Hz. With the eigenvalue of the forming board now significantly higher than the operating frequency of the mill, resonance is impossible and the system is dynamically stable. Additionally, the thinner plates (9.5mm instead of 25mm) meant it was

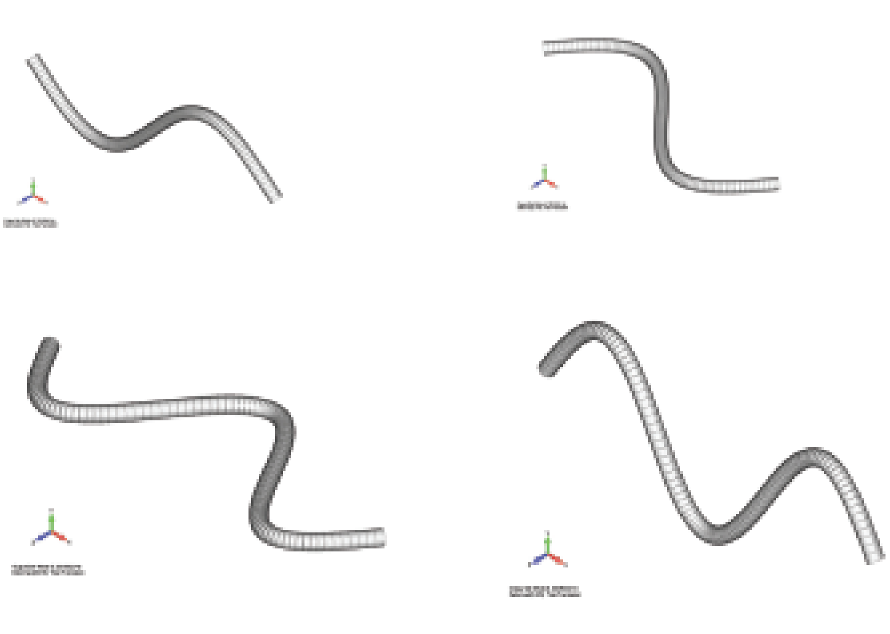

Figure 4: Second and third vibration modal-shape pairs for a simple supported beam.

more than half the cost of the first, marginal redesign.

DYNAMIC LOAD CONSIDERATIONS

When a structure is loaded in a transient or time-varying fashion (e.g., when an electric motor creates a constant, sinusoidally varying load), if the eigenvalue of the structure is lower or higher than the excitation frequency, the structure will just behave as if the load was applied statically. Let us say that we have this structure with an eigenvalue at 10Hz and it is whacked by a transient (e.g., half sinewave with frequency of 10Hz), we would expect the structure to vibrate subsequent to the hit and then gradually return to its static zero-stress condition.

However, if the structure's dynamic load is time-varying (e.g., sine wave at 10Hz), the structure will resonate. If little damping is present (think metal or stiff plastic structures), then we may see the classic harmonic resonance that caused the collapse of the Tacoma Narrows Bridge in 1940. What kills structures is resonance, and the worst kind of resonance occurs when the structure sees the excitation load over and over again. The most effective way to eliminate this worry is to design your structure to have lower or higher natural frequencies than its operational frequency; this goal is the dominant reason for performing an eigen analysis.

THE FINAL MATH

In our prior discussion we haven't mentioned anything about the magnitude of an eigenmode. That is to say, we have discussed its frequency and its shape but left out any description of its magnitude. In eigen analysis (normal mode analysis) no load is applied to the structure. Without a load (e.g., a force or pressure), a prediction of the actual eigenmode is impossible. The extraction of the eigenmode (the shape of the permissible deformation mode) involves a fancy piece of math that is commonly available in a multitude of textbooks. The core thought is that we are solving the dynamic equation:

$$\{f(t)\} = [m]\{x''(t)\} + [C]\{x'(t)\} + [K]\{x(t)\}$$

If damping [C] is ignored (a good assumption for a lot of designs) and the applied force *f(t)* is set to 0.0, the equation reduces to this more manageable formula:

$$[m]\{x''(t)\} + [K]\{x(t)\} = 0$$

This is the key equation for eigen analysis and states that only the mass and the stiffness of the structure control its natural modes.

To solve this equation see your favorite math handbook. The gist of the discussion is that the eigenvalue of the structure boils down into this elegant formula:

$$\omega = \sqrt{K/m}$$

And since no forces are used in the calculation of the eigenvalue, its associated eigenmode is dimensionless. Your FEA program then scales the eigenmode such that the maximum displacement within each mode shape is near 1.0 or some relative value tied to the mass of the structure. When these eigenmodes are dis-

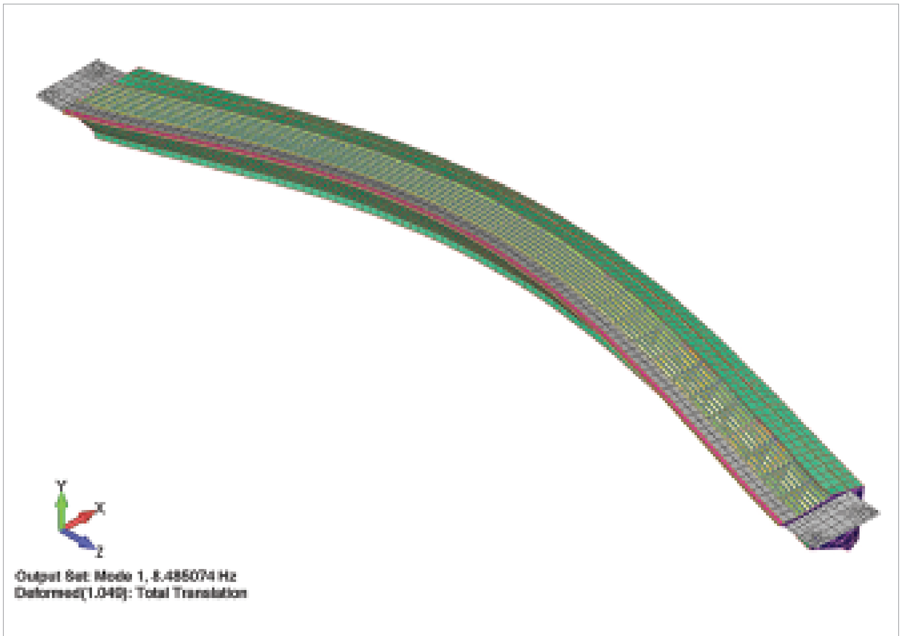

Figure 5: Mode 1 of vibrating paper-mill forming board

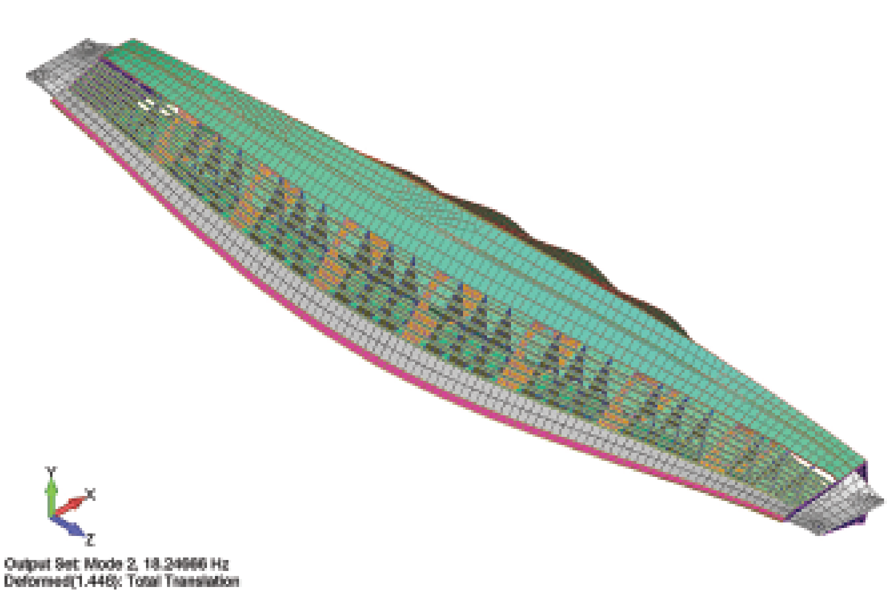

Figure 6: Mode 2 of vibrating paper-mill forming board

played within an FEA program, we see an imaginary magnitude; this visual can be problematic for many initiates who are first venturing into the eigen world of dynamic analysis, but we will discuss the implications in Part II of this series.

MODE ANALYSIS ESSENTIAL CHECKLIST

Determine what type of loading you may have on your structure and whether or not that loading might set up a resonant condition. Try to determine your loading frequencies and ensure that they fall outside of the eigenvalues of your structure.

Run an eigen analysis and look at the first three normal mode frequencies. See if they fall within your danger zone.

If the normal mode frequencies are outside your loading frequencies then stop. You are done and all is good.

If your normal mode frequencies are within your range of interest and you can't redesign around them, then stay tuned for our future articles. We will show that maybe it isn't that bad after all. ■

George Laird, *Ph.D., P.E. is a mechanical engineer with PredictiveEngineering.com and can be reached at FEA@PredictiveEngineering.com. Send your comments about this article to DE-Editors@deskeng.com.*